Some weighted estimates for Littlewood-Paley functions and radial multipliers

著者	Sato Shuichi
journal or publication title	Journal of Mathematical Analysis and Applications
volume	278
number	2
page range	308-323
year	2003-02-15
URL	http://hdl.handle.net/2297/24666

doi: 10.1016/S0022-247X(02)00393-1

SOME WEIGHTED ESTIMATES FOR LITTLEWOOD-PALEY FUNCTIONS AND RADIAL MULTIPLIERS

SHUICHI SATO

ABSTRACT. We prove some weighted estimates for certain Littlewood-Paley operators on the weighted Hardy spaces H^p_w $(0 < p \le 1)$ and on the weighted L^p spaces. We also prove some weighted estimates for the Bochner-Riesz operators and the spherical means.

1. INTRODUCTION

Let $n \ge 2$ and $\rho(x) \in C^\infty(\mathbb{R}^n \setminus \{0\})$ be positive and homogeneous of degree 1. We assume $\nabla\rho \neq 0$ and the hypersurface

$$\Sigma = \{x \in \mathbb{R}^n : \rho(x) = 1\}$$

has non-vannishing Gaussian curvature. We define

$$\sigma_\delta(f)(x) = \left(\int_0^\infty \left|S_R^\delta(f)(x) - S_R^{\delta-1}(f)(x)\right|^2 \frac{dR}{R}\right)^{1/2}, \tag{1.1}$$

where

$$S_R^\delta(f)(x) = \int_{\mathbb{R}^n} \left(1 - R^{-2}\rho(\xi)^2\right)_+^\delta \hat{f}(\xi) e^{2\pi i x\xi}\, d\xi \tag{1.2}$$

is the Bochner-Riesz means of order δ on $\mathbb{R}^n$ with respect to ρ. By Sogge [18] we are motivated to consider $S_R^\delta(f)$ with $\rho(\xi)$ in place of the Euclidean norm $|\xi|$. We also define

$$\tau_\delta(f)(x) = \left(\int_0^\infty \left|\tilde{S}_R^{\delta-1}(f)(x)\right|^2 \frac{dR}{R}\right)^{1/2} \tag{1.3}$$

with

$$\tilde{S}_R^\delta(f)(x) = \int_{\mathbb{R}^n} \eta(\rho(\xi)/R) \left(1 - R^{-2}\rho(\xi)^2\right)_+^\delta \hat{f}(\xi) e^{2\pi i x\xi}\, d\xi, \tag{1.4}$$

where $\eta \in C^\infty(\mathbb{R})$ is such that $\eta(t) = 1$ if $|t| \ge 1/4$ and $\eta(t) = 0$ if $|t| \le 1/8$.

Put $\delta(p) = n|1/p - 1/2| + 1/2$. We first study the behavior of τ_δ, $\delta \ge \delta(p)$, $\delta > \delta(1)$, on the weighted Hardy space $H^p_w(\mathbb{R}^n)$, $0 < p \le 1$. Under these conditions of δ we can write $\tau_\delta(f) = g_\psi(f)$, where $g_\psi(f)$ is the Littlewood-Paley function defined by

$$g_\psi(f)(x) = \left(\int_0^\infty |\psi_t * f(x)|^2 \frac{dt}{t}\right)^{1/2};$$

here $\psi_t(x) = t^{-n}\psi(t^{-1}x)$, and ψ satisfies $|\psi(x)| \le c(1 + |x|)^{-n-\epsilon}$ with $\epsilon = n(1/p - 1) + \delta - \delta(p) > 0$ and $\int_{\mathbb{R}^n} \psi(x)\, dx = 0$. So τ_δ is bounded on the weighted Lebesgue

2000 *Mathematics Subject Classification.* Primary 42B08, 42B10, 42B15, 42B25, 42B30.

spaces L^r_w for all $r \in (1,\infty)$ and all $w \in A_r$ (see Sato [16] and Ding-Fan-Pan [7]), where we denote by A_r the weight class of Muckenhoupt.

Remark 1. We consider $\tilde{S}^\delta_R$ to eliminate the singularity of $\rho(\xi)$ at the origin. If $\rho(\xi) = |\xi|$, this is not needed. For example, we can treat τ_δ and σ_δ in the same way in proving the estimates like those of Theorem 1 when $\rho(\xi) = |\xi|$.

Now we recall the definition of the weighted Hardy spase H^p_w. We begin by defining the weight classes. Let $B(x_0, s)$ be a closed ball of $\mathbb{R}^n$ with center x_0 and radius $s > 0$. Let $w(x)$ be a positive measurable function on $\mathbb{R}^n$. Then we say $w \in B_p$ $(1 < p < \infty)$ if

$$\int_{\mathbb{R}^n} \left|M(\chi_{B(x_0,s)})(x)\right|^p w(x)\, dx \le C_{p,w} w\left(B(x_0,s)\right),$$

where M is the Hardy-Littlewood maximal operator, $w(E) = \int_E w(x)\, dx$ and $C_{p,w}$ is a constant independent of x_0 and s; and we say $w \in B_1$ if

$$\sup_{\lambda>0} \lambda w\left(\left\{x \in \mathbb{R}^n : M(\chi_{B(x_0,s)})(x) > \lambda\right\}\right) \le C_{1,w} w(B(x_0,s)),$$

where $C_{1,w}$ is independent of x_0 and s. Note that $M(\chi_{B(x_0,s)})(x) \approx s^n(s + |x - x_0|)^{-n}$. It is easy to see that $B_r \subset B_p$ for $1 \le r \le p$ and $A_p \subset B_p$ for $1 \le p < \infty$. Also for any $1 < p < \infty$ there exists $w \in B_p$ which does not belong to A_∞ (see [10] and [23]). We observe that if $w \in B_p$ and $t \ge 1$, then

$$w\left(B(x_0, ts)\right) \le Ct^{np} w\left(B(x_0,s)\right).$$

Put $B_\infty = \cup_{p\ge1} B_p$. Choose $\varphi \in \mathcal{S}(\mathbb{R}^n)$ (the Schwartz space) which satisfies $\int_{\mathbb{R}^n} \varphi(x)\, dx = 1$. Let $0 < p \le 1$, $w \in B_\infty$ and let f be a tempered distribution. We say that $f \in H^p_w(\mathbb{R}^n)$ if

$$\|f\|_{H^p_w} = \left(\int_{\mathbb{R}^n} \sup_{t>0} |\varphi_t * f(x)|^p\, w(x)\, dx\right)^{1/p} < \infty.$$

It is convenient to consider a dense subspace of H^p_w. Let $f \in \mathcal{S}(\mathbb{R}^n)$; we say $f \in \mathcal{S}_0(\mathbb{R}^n)$ if its Fourier transform $\hat{f}$ is compactly supported and vanishes in a neighborhood of the origin. It is known that if $0 < p \le 1$ and $w \in B_\infty$, the space $\mathcal{S}_0$ is dense in H^p_w (see [24]).

Also let $L^{p,\infty}_w(\mathbb{R}^n)$ denotes the weighted weak L^p space of all those measurable functions f which satisfy

$$\sup_{\lambda>0} \lambda^p w\left(\{x \in \mathbb{R}^n : |f(x)| > \lambda\}\right) = \|f\|^p_{L^{p,\infty}_w} < \infty.$$

Then we prove the following:

Theorem 1. *Let τ_δ be as in* (1.3).

(1) *Let $0 < p < 1$. Suppose $w \in B_1$ and $w \in A_\infty$. Then*

$$\|\tau_{\delta(p)}(f)\|_{L^{p,\infty}_w} \le C_{p,w} \|f\|_{H^p_w}, \qquad f \in \mathcal{S}_0(\mathbb{R}^n).$$

(2) *Let $0 < p \le 1$ and $\delta > \delta(p)$. Suppose $w \in B_{1+n^{-1}p(\delta-\delta(p))}$ and $w \in A_\infty$. Then*

$$\|\tau_\delta(f)\|_{L^p_w} \le C_{p,\delta,w} \|f\|_{H^p_w}, \qquad f \in \mathcal{S}_0(\mathbb{R}^n).$$

When $\rho(\xi) = |\xi|$, these results also hold for σ_δ in place of τ_δ, as we mentioned in Remark 1. We note that when $\rho(\xi) = |\xi|$ and $w(x) \equiv 1$, Theorem 1 (with σ_δ in place of τ_δ) is due to Kaneko-Sunouchi [12]. By part (1) the Littlewood-Paley operator $\tau_{\delta(p)}$, initially defined on $\mathcal{S}_0$, has a unique sublinear extension which is bounded from H^p_w to $L^{p,\infty}_w$; and by part (2) τ_δ extends likewise to a bounded operator from H^p_w to L^p_w. As for a recent article dealing with the boundedness on the Hardy spaces for the Littlewood-Paley functions, see also Ding-Lu-Xue [8], where they study the Marcinkiewicz integrals.

Remark 2. For a bounded function m define a multiplier operator T_m by $\widehat{(T_m f)}(\xi) = m(\rho(\xi))\hat{f}(\xi)$ and a maximal function $T^*_m f(x) = \sup_{t>0} |T_t f(x)|$, where $\widehat{T_t f}(\xi) = m(t\rho(\xi))\hat{f}(\xi)$. Then by the methods of Carbery [3] (see also [5]) and essentially by Theorem 1 we can prove some estimates for T_m and T^*_m on H^p_w under certain, suitable conditions on m.

We also prove the following weighted L^2 estimates for σ_δ defined in (1.1).

Theorem 2. *If $\delta > 1/2$ and $0 \le \alpha < 1$, then*

$$\int_{\mathbb{R}^n} |\sigma_\delta(f)(x)|^2 |x|^{-\alpha}\, dx \le C_{\delta,\alpha} \int_{\mathbb{R}^n} |f(x)|^2 |x|^{-\alpha}\, dx.$$

In Carbery-Rubio de Francia-Vega [6] this is proved for the case $\rho(\xi) = |\xi|$ (see also Rubio de Francia [14] for another proof). We prove Theorem 2 for the general $\rho(\xi)$ by applying the method of Rubio de Francia [14]. Let S^δ_R be as in (1.2) and define

$$S^\delta_*(f)(x) = \sup_{R>0} |S^\delta_R(f)(x)| \ . \tag{1.5}$$

Then Theorem 2 implies, as in the case $\rho(\xi) = |\xi|$, the following (see [6], [14]):

Corollary 1. *Let $0 < \lambda \le (n-1)/2$. If $-2\lambda - 1 < \alpha < 2n\lambda/(n-1)$, then*

$$\int_{\mathbb{R}^n} \left|S^\lambda_*(f)(x)\right|^2 |x|^\alpha\, dx \le C_{\lambda,\alpha} \int_{\mathbb{R}^n} |f(x)|^2 |x|^\alpha\, dx.$$

As in [6], by Corollary 1 we see that $\lim_{R\to\infty} S^\lambda_R(f)(x) = f(x)$ a.e. for all $\lambda > 0$ and $f \in L^p(\mathbb{R}^n)$ provided $2 \le p < 2n/(n-1-2\lambda)$ (for the case $p < 2$ see Tao [25]).

We can also consider the spherical means with respect to ρ. For $\beta > 0$ let

$$M^\beta_t(f)(x) = c_\beta t^{-n} \int_{\rho(y)<t} (1 - t^{-2}\rho(y)^2)^{\beta-1} f(x-y)\, dy \qquad (f \in \mathcal{S})\,, \tag{1.6}$$

where $c_\beta = \Gamma(\beta + n/2)/(\pi^{n/2}\Gamma(\beta))$. In Section 4 we shall prove some weighted estimates for a modified version of $M^\beta_t(f)$.

We assume $\rho(x) = |x|$ in (1.6) for the rest of this section. By taking the Fourier transform, we can embed these operators in an analytic family of operators in β in such a way that

$$M^0_t(f)(x) = c \int_{S^{n-1}} f(x - ty)\, d\sigma(y),$$

where $d\sigma$ denotes the Lebesgue surface measure on the unit sphere S^{n-1}. We also define $M^\beta_*(f)(x) = \sup_{t>0} \left|M^\beta_t(f)(x)\right|$. The operator M^β_t was studied in Stein [19] (see also Stein-Wainger [21] and Kaneko-Sunouchi [12]).

Now we see some applications of Theorems 1 and 2 to the spherical means.

Remark 3. Define, when $\beta + n/2 - 1 > 0$,

$$\nu_\beta(f)(x) = \left(\int_0^\infty \left| \frac{\partial}{\partial t} \, M_t^\beta(f)(x) \right|^2 t \, dt \right)^{1/2}$$
$$= 2 \left| \beta + \frac{n}{2} - 1 \right| \left(\int_0^\infty |M_t^\beta(f)(x) - M_t^{\beta-1}(f)(x)|^2 \, \frac{dt}{t} \right)^{1/2}.$$

If $\delta = \beta + n/2 - 1 > 0$, then $\sigma_\delta(f)$ and $\nu_\beta(f)$ $(f \in \mathcal{S})$ are pointwise equivalent; that is, there are two positive constants A and B such that

$$\sigma_\delta(f)(x) \le A \, \nu_\beta(f)(x) \le B \, \sigma_\delta(f)(x). \tag{1.7}$$

This was proved by [12]. By (1.7) we immediately get the $\nu_\beta(f)$ analogue of Theorem 1 (see the remark below Theorem 1).

Remark 4. Let $\beta > 3/2 - n/2$ and $0 \le \alpha < 1$. By Theorem 2 for $\rho(\xi) = |\xi|$ (a result of Carbery-Rubio de Francia-Vega [6]) and (1.7) we have

$$\int_{\mathbb{R}^n} |\nu_\beta(f)(x)|^2 |x|^{-\alpha} \, dx \le C_{\beta,\alpha} \int_{\mathbb{R}^n} |f(x)|^2 |x|^{-\alpha} \, dx.$$

Remark 5. We write

$$\mathcal{M}(f)(x) = \sup_{t>0} \left| \int_{S^{n-1}} f(x - ty) \, d\sigma(y) \right|.$$

Note that $\mathcal{M}(f)(x) = c M_*^0(f)(x)$. Let $n \ge 2$, $n/(n-1) < p$. Then Duoandikoetxea-Vega [9] proved that the inequality

$$\int_{\mathbb{R}^n} |\mathcal{M}(f)(x)|^p |x|^{-\alpha} \, dx \le C \int_{\mathbb{R}^n} |f(x)|^p |x|^{-\alpha} \, dx \tag{1.8}$$

holds for $n - p(n-1) < \alpha < n - 1$ (this was partly proved in Rubio de Francia [13]) and does not hold for $\alpha > n - 1$. Stein [19] proved (1.8) when $n \ge 3$, $\alpha = 0$; the result for $\alpha = 0$ and $n = 2$ is due to Bourgain [1] (see also [18]). By Remark 4 and a well-known argument (see [19] and also [21]) we can give another proof of the inequality (1.8) when $n \ge 3$, $0 \le \alpha < n - 1$ and $n/(n-1) < p$.

In the following sections we shall give the proofs of the theorems and the corollary stated above.

2. Proof of Theorem 1

To show Theorem 1 we prove a more general result. For a locally integrable function f, a non-negative integer m and $\sigma \ge 0$, we define

$$|f|_{m,\sigma} = \sup_{z \in \mathbb{R}^n, s \in (0,1]} \inf_{Q \in \mathcal{P}_m} s^{-\sigma-n} \int_{B(z,s)} |f(y) - Q(y)| \, dy,$$

where $\mathcal{P}_m$ denotes the collection of polynomials of degree less than or equal to m. We also write $|f|_{m,\sigma} = |f : m, \sigma|$.

Let $\theta > n$ and let ψ be a measurable function on $\mathbb{R}^n$ satisfying the following properties:

$$|\psi(x)| \le C(1 + |x|)^{-\theta}, \tag{2.1}$$
$$\int_{\mathbb{R}^n} \psi(x) \, dx = 0; \tag{2.2}$$

furthermore, ψ can be written as

$$\psi(x) = \sum_{k=0}^{\infty} 2^{-k\theta}\eta_k(x), \tag{2.3}$$

where $\{\eta_k\}_{k\geq 0}$ is a sequence of integrable functions satisfying the following:

$$\operatorname{supp}(\eta_k) \subset \{2^{k-2} \leq |x| \leq 2^{k+2}\} \ (k \geq 1), \quad \operatorname{supp}(\eta_0) \subset \{|x| \leq 1\}, \tag{2.4}$$

$$\sup_{j\geq 1} |\eta_j : [\theta - n], \theta - n + \kappa| < \infty \ \text{ for some } \kappa > 0, \tag{2.5}$$

$$|\eta_0 : [\theta - n], \theta - n| < \infty. \tag{2.6}$$

Here $[a]$ denotes the greatest integer less than or equal to a. Then we shall prove the following:

Proposition 1. *Let g_ψ be the Littlewood-Paley operator with ψ satisfying* (2.1) *to* (2.6).

(1) *Let $0 < p < 1$. Suppose $\theta = n/p$, $w \in B_1$ and $w \in A_\infty$. Then*
$$\|g_\psi(f)\|_{L^{p,\infty}_w} \leq C_{p,w}\|f\|_{H^p_w}, \qquad f \in \mathcal{S}_0(\mathbb{R}^n).$$

(2) *Let $0 < p \leq 1$. Suppose $\theta > n/p$, $w \in B_{p\theta/n}$ and $w \in A_\infty$. Then*
$$\|g_\psi(f)\|_{L^p_w} \leq C_{p,\theta,w}\|f\|_{H^p_w}, \qquad f \in \mathcal{S}_0(\mathbb{R}^n).$$

To prove Proposition 1 we use the following result:

Proposition 2. *Let $\Psi \in L^1(\mathbb{R}^n)$ satisfy $\int_{\mathbb{R}^n} \Psi(x)\,dx = 0$ and let $\theta > n$. Suppose that*

$$\left(\int_0^\infty \inf_{P\in\mathcal{P}_{[\theta-n]}} \left(\int_{|y|<1} |r^n\Psi(r(x-y)) - P(y)|\,dy\right)^2 \frac{dr}{r}\right)^{1/2} \leq C|x|^{-\theta} \tag{2.7}$$

for $|x| > 2$. Then we have the following:

(1) *Let $0 < p < 1$. Suppose $\theta = n/p$ and $w \in B_1$. If the operator g_Ψ is bounded on $L^{p_0}_w$ for some $p_0 \in (p, \infty)$, then*
$$\|g_\Psi(f)\|_{L^{p,\infty}_w} \leq C_{p,w}\|f\|_{H^p_w}, \qquad f \in \mathcal{S}_0(\mathbb{R}^n).$$

(2) *Let $0 < p \leq 1$. Suppose $\theta > n/p$ and $w \in B_{p\theta/n}$. If the operator g_Ψ is bounded on $L^{p_0}_w$ for some $p_0 \in (p, \infty)$, then*
$$\|g_\Psi(f)\|_{L^p_w} \leq C_{p,\theta,w}\|f\|_{H^p_w}, \qquad f \in \mathcal{S}_0(\mathbb{R}^n).$$

We use the atomic decomposition to prove Proposition 2. Let N be a non-negative integer and w be a locally integrable positive function on $\mathbb{R}^n$. Then a measurable function a on $\mathbb{R}^n$ is called a (p, N, w) atom $(0 < p \leq 1)$ if for some x_0 and s we have

$$\operatorname{supp}(a) \subset B(x_0, s), \tag{2.8}$$

$$\|a\|_\infty \leq w(B(x_0, s))^{-1/p}; \tag{2.9}$$

and

$$\int_{\mathbb{R}^n} a(x)x^\alpha\,dx = 0 \quad \text{for all } |\alpha| \leq N, \tag{2.10}$$

where $\alpha = (\alpha_1, \dots, \alpha_n)$ is a multi-index and $x^\alpha = x_1^{\alpha_1} \dots x_n^{\alpha_n}$, $|\alpha| = \alpha_1 + \dots + \alpha_n$.

Lemma 1. *Let $\Psi \in L^1(\mathbb{R}^n)$ satisfy $\int_{\mathbb{R}^n} \Psi(x)\,dx = 0$ and (2.7).*

(1) *Let $0 < p < 1$. Suppose $\theta = n/p$ and $w \in B_1$. If the operator g_Ψ is bounded on $L^{p_0}_w$ for some $p_0 \in (p, \infty)$, then for a $(p, [n/p - n], w)$ atom a we have*
$$w\left(\{x \in \mathbb{R}^n : g_\Psi(a)(x) > \lambda\}\right) \le C\lambda^{-p},$$
where C is independent of a and λ.

(2) *Let $0 < p \le 1$. Suppose $\theta > n/p$ and $w \in B_{p\theta/n}$. If the operator g_Ψ is bounded on $L^{p_0}_w$ for some $p_0 \in (p, \infty)$, then for a $(p, [\theta - n], w)$ atom a we have*
$$\|g_\Psi(a)\|_{L^p_w} \le C,$$
where C is independent of a.

This follows from the following result:

Lemma 2. *Let $\Psi \in L^1(\mathbb{R}^n)$ satisfy $\int_{\mathbb{R}^n} \Psi(x)\,dx = 0$ and (2.7). Let a be a $(p, [\theta - n], w)$ atom supported in $B(x_0, s)$ with (2.9). Then we have*
$$g_\Psi(a)(x) \le C\left(|B(x_0,s)|/w(B(x_0,s))\right)^{1/p} s^{(\theta - n/p)}(s + |x - x_0|)^{-\theta}$$
for x with $|x - x_0| > 2s$.

Proof. We first give a proof for the case $w(x) \equiv 1$. By (2.7)–(2.10) with $N = [\theta - n]$ we have, if $|x - x_0| > 2s$,
$$\begin{aligned}
&g_\Psi(a)(x)^2 = \int_0^\infty \left|\int_{\mathbb{R}^n} a(y) r^n \Psi(r(x-y))\,dy\right|^2 \frac{dr}{r}\\
&= \int_0^\infty \inf_{P \in \mathcal{P}_{[\theta-n]}} \left|\int_{B(x_0,s)} a(y)\left(r^n\Psi(r(x-y)) - P(y)\right)\,dy\right|^2 \frac{dr}{r}\\
&\le \|a\|_\infty^2 \int_0^\infty \inf_{P \in \mathcal{P}_{[\theta-n]}} \left(\int_{B(x_0,s)} |r^n\Psi(r(x-y)) - P(y)|\,dy\right)^2 \frac{dr}{r}\\
&= \|a\|_\infty^2 \int_0^\infty \inf_{P \in \mathcal{P}_{[\theta-n]}} \left(\int_{|y|<1} \left|(rs)^n\Psi(rs(s^{-1}x - s^{-1}x_0 - y)) - P(y)\right|\,dy\right)^2 \frac{dr}{r}\\
&\le C\|a\|_\infty^2 (s^{-1}|x - x_0|)^{-2\theta}\\
&\le Cs^{-2n/p + 2\theta}|x - x_0|^{-2\theta} \le Cs^{2(\theta - n/p)}(s + |x - x_0|)^{-2\theta}.
\end{aligned}$$

Next, let a be a $(p, [\theta - n], w)$ atom supported in $B(x_0, s)$ with (2.9). Then applying the above estimate to
$$\left(w(B(x_0,s))/|B(x_0,s)|\right)^{1/p} a,$$
we get the conclusion. □

Now we give the proof of Lemma 1. We first prove part (1). Let a be a $(p, [n/p - n], w)$ atom supported in $B(x_0, s)$ with (2.9). Then
$$\begin{aligned}
w\left(\{x \in \mathbb{R}^n : g_\Psi(a)(x) > \lambda\}\right) &\le w\left(\{x \in B(x_0, 2s) : g_\Psi(a)(x) > \lambda\}\right)\\
&\quad + w\left(\{x \in \mathbb{R}^n \setminus B(x_0, 2s) : g_\Psi(a)(x) > \lambda\}\right)\\
&= I + II, \quad \text{say.}
\end{aligned}$$

Since g_Ψ is bounded on $L^{p_0}_w$, by Chebyshev's inequality and Hölder's inequality we have

$$
\begin{aligned}
I &\le \lambda^{-p} \int_{B(x_0,2s)} |g_\Psi(a)(x)|^p w(x)\,dx \\
&\le \lambda^{-p} w\left(B(x_0,2s)\right)^{(p_0-p)/p_0} \left(\int |g_\Psi(a)(x)|^{p_0} w(x)\,dx\right)^{p/p_0} \\
&\le C\lambda^{-p} w\left(B(x_0,2s)\right)^{(p_0-p)/p_0} \left(\int |a(x)|^{p_0} w(x)\,dx\right)^{p/p_0} \\
&\le C\lambda^{-p} w\left(B(x_0,2s)\right)^{(p_0-p)/p_0} w\left(B(x_0,2s)\right)^{-1+p/p_0} \\
&= C\lambda^{-p},
\end{aligned}
\tag{2.11}
$$

where to get the last inequality we have used the doubling condition.

Next, by Lemma 2 we see that

$$
\begin{aligned}
II &\le w\left(\left\{x \in \mathbb{R}^n : C\left(|B(x_0,s)|/w(B(x_0,s))\right)^{1/p} (s+|x-x_0|)^{-n/p} > \lambda\right\}\right) \\
&= w\left(\left\{x \in \mathbb{R}^n : Cs^n(s+|x-x_0|)^{-n} > w(B(x_0,s))\lambda^p\right\}\right) \\
&= III, \quad \text{say.}
\end{aligned}
$$

Since $w \in B_1$, recalling that $s^n(s+|x-x_0|)^{-n} \approx M(\chi_{B(x_0,s)})(x)$, we have

$$III \le w\left(\left\{x \in \mathbb{R}^n : M(\chi_{B(x_0,s)})(x) > w(B(x_0,s))\lambda^p\right\}\right) \le C\lambda^{-p}.$$

Combining the estimates for I and II, we conclude the proof of part (1).

Next we turn to the proof of part (2). Let a be a $(p,[\theta-n],w)$ atom supported in $B(x_0,s)$ with (2.9). Then by Lemma 2 we have

$$g_\Psi(a)(x) \le Cw(B(x_0,s))^{-1/p} M(\chi_{B(x_0,s)})(x)^{\theta/n} \qquad \text{for } |x-x_0| > 2s.$$

Since $w \in B_{p\theta/n}$, we find

$$
\begin{aligned}
\int_{\mathbb{R}^n\setminus B(x_0,2s)} g_\Psi(a)(x)^p w(x)\,dx &\le Cw(B(x_0,s))^{-1} \int_{\mathbb{R}^n} M(\chi_{B(x_0,s)})(x)^{p\theta/n} w(x)\,dx \\
&\le C.
\end{aligned}
$$

Combining this with the estimate appearing in (2.11), we get the conclusion.

To prove Proposition 2 (1) we need the following result (see [20]):

Lemma 3. *Let $0 < p < 1$. Suppose $\{f_k\}$ is a sequence of measurable functions on $\mathbb{R}^n$ such that*

$$\sup_{\lambda>0} \lambda^p w\left(\{x : |f_k(x)| > \lambda\}\right) \le 1 \quad \textit{for all } k,$$

and suppose $\{c_k\}$ is a sequence of complex numbers satisfying $\sum |c_k|^p \le 1$. Then we have

$$\sup_{\lambda>0} \lambda^p w\left(\left\{x \in \mathbb{R}^n : \sum |c_k f_k(x)| > \lambda\right\}\right) \le \frac{2-p}{1-p}.$$

Now we can prove Proposition 2. We note that $f \in \mathcal{S}_0(\mathbb{R}^n)$ can be decomposed as $f = \sum \lambda_k a_k$ by $(p,[\theta-n],w)$-atoms ($w \in B_{p\theta/n}$), where we have $\sum \lambda_k^p \le C\|f\|^p_{H^p_w}$, $\sum \lambda_k a_k = f$ a.e. and $\sum \lambda_k |a_k| \le Cf^*$, with f^* denoting the grand maximal function (see [24]). Using this decomposition, we first prove part (1). Since f^* is bounded, by the dominated convergence theorem we have $\Psi_t * f = \sum \lambda_k \Psi_t * a_k$ a.e. and so $g_\Psi(f) \le \sum_k |\lambda_k| g_\Psi(a_k)$. Thus by Lemma 1 (1) and Lemma 3 we see that

$$\sup_{\lambda>0} \lambda^p w\left(\{x \in \mathbb{R}^n : g_\Psi(f)(x) > \lambda\}\right) \le C \sum \lambda_k^p \le C\|f\|_{H_w^p}^p.$$

This completes the proof of Proposition 2 (1). Part (2) can be proved in the same way by using Lemma 1 (2).

Now we turn to the proof of Proposition 1. First we see that if ψ satisfies the conditions (2.1)–(2.6), then ψ satisfies the condition (2.7) of Proposition 2. Let $|x| > 2$. Then by (2.1) we have

$$\begin{aligned} \int_1^\infty \left(\int_{|y|<1} |r^n \psi(r(x-y))|\,dy\right)^2 \frac{dr}{r} &\le C \int_1^\infty r^{2n}(1+r|x|)^{-2\theta}\,\frac{dr}{r} \\ &\le C|x|^{-2\theta} \int_1^\infty r^{2n-2\theta}\,\frac{dr}{r} \le C|x|^{-2\theta}. \end{aligned} \tag{2.12}$$

Let $r \le 1$. Suppose $2^m|x|^{-1} \le r < 2^{m+1}|x|^{-1}$ for $m \le m_x := [(\log 2)^{-1}\log|x|]$. If $|y| \le 1$, then $r|x|/2 \le r|x-y| \le 3r|x|/2$. Therefore, if $m \ge 5$, by (2.3) and (2.4) we have

$$\psi(r(x-y)) = \sum_{k=m-3}^{m+5} 2^{-k\theta}\eta_k(r(x-y)).$$

This expression of ψ and (2.5) imply that there exists a polynomial $P = P_{r,x} \in \mathcal{P}_{[\theta-n]}$ such that

$$\int_{|y|<1} |r^n\psi(r(x-y)) - P(y)|\,dy \le Cr^{\kappa+\theta}2^{-m\theta} \le C|x|^{-\kappa-\theta}2^{m\kappa}. \tag{2.13}$$

If $m \le 4$, then

$$\psi(r(x-y)) = \sum_{k=0}^{8} 2^{-k\theta}\eta_k(r(x-y)).$$

Therefore, by (2.5) and (2.6) there exists a polynomial $P = P_{r,x} \in \mathcal{P}_{[\theta-n]}$ such that

$$\int_{|y|<1} |r^n\psi(r(x-y)) - P(y)|\,dy \le Cr^{\theta} \le C|x|^{-\theta}2^{m\theta}. \tag{2.14}$$

By (2.13) and (2.14) we have

$$\begin{aligned} &\int_0^1 \inf_{P\in\mathcal{P}_{[\theta-n]}} \left(\int_{|y|<1} |r^n\psi(r(x-y)) - P(y)|\,dy\right)^2 \frac{dr}{r} \\ &\le \sum_{m\le m_x} \int_{2^m|x|^{-1}}^{2^{m+1}|x|^{-1}} \inf_{P\in\mathcal{P}_{[\theta-n]}} \left(\int_{|y|<1} |r^n\psi(r(x-y)) - P(y)|\,dy\right)^2 \frac{dr}{r} \\ &\le \sum_{m\le 4} C|x|^{-2\theta}2^{2m\theta} + \sum_{5\le m\le m_x} C|x|^{-2(\kappa+\theta)}2^{2m\kappa} \le C|x|^{-2\theta}. \end{aligned} \tag{2.15}$$

Now the condition (2.7) of Proposition 2 follows from (2.12) and (2.15).

Also by [16] we see that the conditions (2.1) and (2.2) imply the L^p_w-boundedness of g_ψ for all $p \in (1,\infty)$ and all $w \in A_p$. So Proposition 1 follows from Proposition 2.

Now we give the proof of Theorem 1. Let

$$K^{\delta}(x)=\int_{\mathbb{R}^n}\eta(\rho(\xi))\left(1-\rho(\xi)^2\right)_+^{\delta}e^{2\pi i x\xi}\,d\xi.$$

Then

$$\left|D^{\alpha}K^{\delta-1}(x)\right|\leq C_{\alpha}(1+|x|)^{-\delta-(n-1)/2} \tag{2.16}$$

for all α, where $D^{\alpha}=(\partial/\partial x_1)^{\alpha_1}\dots(\partial/\partial x_n)^{\alpha_n}$ (see [18]). Therefore, by [15] we see that $K^{\delta-1}$ satisfies the conditions (2.1)–(2.6) for ψ with $\theta=\delta+(n-1)/2$ and $0<\kappa\leq[\delta-(n+1)/2]+1-\delta+(n+1)/2$ in (2.5). Thus Theorem 1 follows from Proposition 1.

3. Proofs of Theorem 2 and Corollary 1

The following result can be used to prove Theorem 2.

Proposition 3. *Let $0<\delta<1$ and suppose that $m_\delta(r)=\chi_{[1-\delta,1]}(r)$ or $m_\delta(r)$ is a continuously differentiable function supported in the interval $[1-\delta,1]$ and satisfying $\|(d/dr)m_\delta\|_{L^1(\mathbb{R})}\leq 1$. Define*

$$\widehat{\left(U_t^{\delta}f\right)}(\xi)=\hat{f}(\xi)m_\delta(t\rho(\xi)).$$

Then for $0\leq\alpha<1$ we have

$$\int_{\mathbb{R}^n}\int_0^{\infty}\left|U_t^{\delta}f(x)\right|^2|x|^{-\alpha}\,\frac{dt}{t}\,dx\leq C_\alpha\delta\int_{\mathbb{R}^n}|f(x)|^2|x|^{-\alpha}\,dx,$$

where C_α is independent of δ.

This was proved in Carbery-Rubio de Francia-Vega [6] and Rubio de Francia [14] when $\rho(\xi)=|\xi|$. To prove the general case we use the method of [14], which is based on an application of Hirschman's method in [11] and the weighted estimates for the one dimensional square functions. To apply that method to our case we only need to observe that $\Lambda(x)=(\|x\|/\rho(x))x$ is bi-Lipschitz, with $\|x\|=\max(|x_1|,\dots,|x_n|)$, that is

$$A|x-y|\leq|\Lambda(x)-\Lambda(y)|\leq B|x-y|$$

for some constants A, $B>0$; but this is an easy consequence of the fact that $\rho(x)$ is positive, homogeneous of degree one and C^{∞} in $\mathbb{R}^n\setminus\{0\}$.

Now we prove Theorem 2. We decompose

$$\rho(\xi)^2(1-\rho(\xi)^2)_+^{\delta-1}=\sum_{k=0}^{\infty}2^{-(\delta-1)k}m_k(\rho(\xi)),$$

where $m_k(t)\in C_0^{\infty}(\mathbb{R})$, $\operatorname{supp}(m_k)\subset[1-2^{-k},1]$ and $|(d/dr)m_k(r)|\leq C2^k$, for $k\geq 1$. Put $\psi_k(x)=\mathcal{F}^{-1}(m_k(\rho(\xi)))(x)$ and $g_k(f)=g_{\psi_k}(f)$, where $\mathcal{F}^{-1}$ denotes the inverse Fourier transform. We can take $m_0(t)$ so that g_0 is bounded on L^2_w for any $w\in A_2$. Now by Proposition 3 for $k\geq 1$ we have

$$\|g_k(f)\|_{L^2(|x|^{-\alpha})}\leq C2^{-k/2}\|f\|_{L^2(|x|^{-\alpha})}\qquad\text{for }0\leq\alpha<1.$$

Thus if $\delta > 1/2$ we have

$$\begin{aligned}\|\sigma_\delta(f)\|_{L^2(|x|^{-\alpha})} &\le \sum_{k=0}^{\infty} 2^{-(\delta-1)k}\|g_k(f)\|_{L^2(|x|^{-\alpha})} \\ &\le \sum_{k=0}^{\infty} C2^{-(\delta-1)k}2^{-k/2}\|f\|_{L^2(|x|^{-\alpha})} \\ &\le C_\delta\|f\|_{L^2(|x|^{-\alpha})}.\end{aligned}$$

This completes the proof.

To apply the result to the maximal operator S_*^δ defined in (1.5) we use the following, which can be proved as in the case $\rho(\xi) = |\xi|$ (see Stein-Weiss [22, Chap. VII]).

Lemma 4. *Let S_R^δ be as in* (1.2)*. If $\beta > 0$ and $\delta > -1$, then we have*

$$S_R^{\delta+\beta}(f)(x) = \frac{2\Gamma(\delta+\beta+1)}{\Gamma(\delta+1)\Gamma(\beta)}\int_0^1 (1-t^2)^{\beta-1}t^{2\delta+1}S_{Rt}^\delta(f)(x)\,dt,$$

for a suitable function f.

Here we give the proof of Corollary 1. Using Lemma 4 and Theorem 2 and arguing as in the proof of [22, Lemma 5.10] we have

$$\|S_*^\lambda(f)\|_{L^2(|x|^\alpha)} \le C_{\lambda,\alpha}\|f\|_{L^2(|x|^\alpha)} \tag{3.1}$$

for all $\lambda > 0$ and $-1 < \alpha \le 0$. It is known that if $\lambda \ge (n-1)/2$, then

$$\|S_*^\lambda(f)\|_{L^2(|x|^\beta)} \le C_{\lambda,\beta}\|f\|_{L^2(|x|^\beta)} \tag{3.2}$$

for $-n < \beta < n$. We extend the estimates (3.1) and (3.2) to complex λ and interpolating between them, we get the conclusion.

Remark 6. Let

$$H_\delta(f)(x) = \left(\int_0^\infty \left|s_R^\delta(f)(x) - s_R^{\delta-1}(f)(x)\right|^2 \frac{dR}{R}\right)^{1/2},$$

where

$$s_R^\delta(f)(x) = \int_{\mathbb{R}^n} \left(1-R^{-1}\rho(\xi)\right)_+^\delta \hat{f}(\xi)e^{2\pi i x\xi}\,d\xi.$$

Then we can prove the weighted estimates of Theorem 2 for H_δ in place of σ_δ by the same argument as in the case of σ_δ. This result also can be used to prove the estimate (3.1) (see [2, 3, 4]).

4. Further results

For a locally integrable function f, a non-negative integer m and $\sigma \ge 0$, we define

$$|f|_{m,\sigma}^* = \sup_{z\in\mathbb{R}^n, s>0}\inf_{Q\in\mathcal{P}_m} s^{-\sigma-n}\int_{B(z,s)}|f(y)-Q(y)|\,dy.$$

Let $\psi \in L^1(\mathbb{R}^n)$ and $\theta \ge 0$. We say $\psi \in \mathcal{F}(m,\sigma,\theta)$ if ψ can be written as in (2.3) with $\{\eta_k\}_{k\ge0}$ satisfying (2.4) and the condition $\sup_{k\ge0}|\eta_k|_{m,\sigma}^* < \infty$. This function class was introduced by Sato [15] to make a unified approach to the studies of maximal Bochner-Riesz means and maximal spherical means in certain problems. By the methods in the proof of Theorem 1 we can prove the following:

Proposition 4. *Let $\theta > n$ and $L \in \mathcal{F}([\theta - n], \theta - n, \theta)$. Define $T^*(f)(x) = \sup_{t>0} |L_t * f(x)|$.*

(1) *Let $0 < p < 1$. Suppose $\theta = n/p$ and $w \in B_1$. Then*

$$\|T^*(f)\|_{L^{p,\infty}_w} \leq C_{p,w}\|f\|_{H^p_w}, \qquad f \in \mathcal{S}_0(\mathbb{R}^n).$$

(2) *Let $0 < p \leq 1$. Suppose $\theta > n/p$ and $w \in B_{p\theta/n}$. Then*

$$\|T^*(f)\|_{L^p_w} \leq C_{p,\theta,w}\|f\|_{H^p_w}, \qquad f \in \mathcal{S}_0(\mathbb{R}^n).$$

(3) *Let $0 < p \leq 1$. Suppose $\theta > n/p$, $w \in B_{p\theta/n}$ and $w \in A_\infty$. Then*

$$\|L_t * f\|_{H^p_w} \leq C_{p,\theta,w}\|f\|_{H^p_w}, \qquad f \in \mathcal{S}_0(\mathbb{R}^n),$$

where the constant $C_{p,\theta,w}$ is independent of $t > 0$.

Proof. Since $L \in \mathcal{F}([\theta - n], \theta - n, \theta)$, arguing as in [15] we have

$$\begin{aligned} T^*(a)(x) &\leq C\left(|B(x_0, s)|/w(B(x_0, s))\right)^{1/p} s^{(\theta - n/p)}(s + |x - x_0|)^{-\theta} \\ &\leq Cw(B(x_0, s))^{-1/p} M(\chi_{B(x_0,s)})(x)^{\theta/n}, \end{aligned} \tag{4.1}$$

where a is a $(p, [\theta - n], w)$ atom supported in $B(x_0, s)$ with (2.9). As in the case of the proof of Proposition 2, this implies parts (1) and (2). Part (3) follows from this estimate along with the multiplier characterization of the weighted Hardy spaces (see [24, Chap. VI, Theorem 4]), which requires the condition $w \in A_\infty$. This completes the proof. □

When $w \in A_1$, part (1) of Proposition 4 is in [15]. Also, if $0 < p < 1$, $w \in A_1$ and $\rho(\xi) = |\xi|$, it is known that $S_*^{\delta(p)-1}$ extends to a bounded operator from H^p_w to $L^{p,\infty}_w$ (see [15]). Let $\theta = \delta + (n-1)/2$, $\delta \geq \delta(p)$, $0 < p \leq 1$, $\delta > \delta(1)$. Then the estimate (2.16) implies that $K^{\delta-1} \in \mathcal{F}([\theta - n], \theta - n, \theta)$ (see [15]). Thus by Proposition 4 we have the following:

Corollary 2. *Let $\tilde{S}^\delta_*(f)(x) = \sup_{R>0} |\tilde{S}^\delta_R(f)(x)|$, where $\tilde{S}^\delta_R(f)(x)$ is as in* (1.4).

(1) *Let $0 < p < 1$ and $w \in B_1$. Then*

$$\|\tilde{S}^{\delta(p)-1}_*(f)\|_{L^{p,\infty}_w} \leq C_{p,w}\|f\|_{H^p_w}, \qquad f \in \mathcal{S}_0(\mathbb{R}^n).$$

(2) *Let $0 < p \leq 1$, $\delta > \delta(p)$ and $w \in B_{1+n^{-1}p(\delta-\delta(p))}$. Then*

$$\left\|\tilde{S}^{\delta-1}_*(f)\right\|_{L^p_w} \leq C_{p,\delta,w}\|f\|_{H^p_w}, \qquad f \in \mathcal{S}_0(\mathbb{R}^n).$$

(3) *Let $0 < p \leq 1$, $\delta > \delta(p)$, $w \in B_{1+n^{-1}p(\delta-\delta(p))}$ and $w \in A_\infty$. Then*

$$\left\|\tilde{S}^{\delta-1}_R(f)\right\|_{H^p_w} \leq C_{p,\delta,w}\|f\|_{H^p_w}, \qquad f \in \mathcal{S}_0(\mathbb{R}^n),$$

where the constant $C_{p,\delta,w}$ is independent of $R > 0$.

Part (3) of Corollary 2 extends a result of Sjölin [17] to the weighted Hardy spaces. When $\rho(\xi) = |\xi|$ and $w(x) \equiv 1$, part (1) (with $S^\delta_*(f)$ in place of $\tilde{S}^\delta_*(f)$) is proved in Stein-Taibleson-Weiss [20]. The estimate for $\tilde{S}^\delta_*$ similar to [20, (2.9)] immediately follows from (2.16), as we can see from the proof of [20, (2.9)]. We can also have the estimate (4.1) for $\tilde{S}^{\delta-1}_*$ in place of T^* as an application of that estimate.

If $0 < p < 1$, $w \in A_1$ and $\rho(x) = |x|$, then it is known that $M_*^{\beta(p)-1/2}$ is bounded from H^p_w to $L^{p,\infty}_w$, where $\beta(p) = n(1/p - 1) + 3/2$ (see [15]). For $\beta > 0$ let

$$\tilde{M}_t^{\beta}(f)(x) = c_\beta t^{-n} \int_{\rho(y)<t} \eta(\rho(y)/t)(1 - t^{-2}\rho(y)^2)^{\beta-1} f(x-y)\, dy,$$

where c_β is as in (1.6) and η is as in (1.4). Then $\eta(\rho(y))(1-\rho(y)^2)_+^{\beta-1} \in \mathcal{F}([\theta - n], \theta - n, \theta)$, where $\beta > 1$ and $\theta = \beta + n - 1$, and hence by Proposition 4 we also have the following:

Corollary 3. *Let* $\tilde{M}_*^{\beta}(f)(x) = \sup_{t>0} |\tilde{M}_t^{\beta}(f)(x)|$ *and write* $\beta(p) = n(1/p - 1) + 3/2$.

(1) *Let* $0 < p < 1$ *and* $w \in B_1$. *Then*

$$\|\tilde{M}_*^{\beta(p)-1/2}(f)\|_{L^{p,\infty}_w} \le C_{p,w}\|f\|_{H^p_w}, \qquad f \in \mathcal{S}_0(\mathbb{R}^n).$$

(2) *Let* $0 < p \le 1$, $\beta > \beta(p)$ *and* $w \in B_{1+n^{-1}p(\beta-\beta(p))}$. *Then*

$$\left\|\tilde{M}_*^{\beta-1/2}(f)\right\|_{L^p_w} \le C_{p,\beta,w}\|f\|_{H^p_w}, \qquad f \in \mathcal{S}_0(\mathbb{R}^n).$$

(3) *Let* $0 < p \le 1$, $\beta > \beta(p)$, $w \in B_{1+n^{-1}p(\beta-\beta(p))}$ *and* $w \in A_\infty$. *Then*

$$\left\|\tilde{M}_t^{\beta-1/2}(f)\right\|_{H^p_w} \le C_{p,\beta,w}\|f\|_{H^p_w}, \qquad f \in \mathcal{S}_0(\mathbb{R}^n),$$

where the constant $C_{p,\beta,w}$ *is independent of* $t > 0$.

When $\rho(\xi) = |\xi|$ and $w(x) \equiv 1$, part (1) of Corollary 3 with $M_*^{\beta}(f)$ in place of $\tilde{M}_*^{\beta}(f)$ is proved in Stein-Taibleson-Weiss [20]. The estimate (4.1) for $\tilde{M}_*^{\beta-1/2}$ in place of T^* also follows from an application of the argument in [20].

Acknowledgments

The author would like to thank the referee for a helpful suggestion.

References

1. J. Bourgain, Averages in the plane over convex curves and maximal operators, J. Analyse Math. 47 (1986), 69–85.
2. A. Carbery, The boundedness of the maximal Bochner-Riesz operator on $L^4(\mathbb{R}^2)$, Duke Math. J. 50 (1983), 409–416.
3. A. Carbery, Radial Fourier multipliers and associated maximal functions, Recent progress in Fourier analysis (I. Peral and J.-L. Rubio de Francia (Editors)) Elsevier science Publishers B. V. (North-Holland) (1985), 49–56.
4. A. Carbery, A weighted inequality for the maximal Bochner-Riesz operator on $\mathbb{R}^2$, Trans. Amer. Math. Soc. 287 (1985), 673–680.
5. A. Carbery, G. Gasper and W. Trebels, Radial Fourier multipliers of $L^p(\mathbb{R}^2)$, Proc. Nat. Acad. Sci. U.S.A. 81 (1984), 3254–3255.
6. A. Carbery, J. L. Rubio de Francia and L. Vega, Almost everywhere summability of Fourier integrals, J. London Math. Soc. (2) 38 (1988), 513–524.
7. Y. Ding, D. Fan and Y. Pan, Weighted boundedness for a class of rough Marcinkiewicz integrals, Indiana Univ. Math. J. 48 (1999), 1037–1055.
8. Y. Ding, S. Lu and Q. Xue, Marcinkiewicz integral on Hardy spaces, Integral Equations Operator Theory 42 (2002), 174–182.
9. J. Duoandikoetxea and L. Vega, Spherical means and weighted inequalities, J. London Math. Soc. (2) 53 (1996), 343–353.
10. C. Fefferman and B. Muckenhoupt, Two nonequivalent conditions for weight functions, Proc. Amer. Math. Soc. 45 (1974), 99–104.
11. I. Hirschman, Multiplier transformations II, Duke Math. J. 28 (1962), 45–56.

12. M. Kaneko and G. Sunouchi, On the Littlewood-Paley and Marcinkiewicz functions in higher dimensions, Tôhoku Math. J. 37 (1985), 343–365.
13. J. L. Rubio de Francia, Weighted norm inequalities for homogeneous families of operators, Trans. Amer. Math. Soc. 275 (1983), 781–790.
14. J. L. Rubio de Francia, Transference principles for radial multipliers, Duke Math. J. 58 (1989), 1–19.
15. S. Sato, Weak type estimates for some maximal operators on the weighted Hardy spaces, Ark. Mat. 33 (1995), 377–384.
16. S. Sato, Remarks on square functions in the Littlewood-Paley theory, Bull. Austral. Math. Soc. 58 (1998), 199–211.
17. P. Sjölin, Convolution with oscillating kernels on H^p spaces, J. London Math. Soc. (2) 23 (1981), 442–454.
18. C. D. Sogge, Fourier integrals in classical analysis, Cambridge University Press, 1993.
19. E. M. Stein, Maximal functions: Spherical means, Proc. Nat. Acad. Sci. U.S.A. 73 (1976), 2174–2175.
20. E. M. Stein, M. H. Taibleson and G. Weiss, Weak type estimates for maximal operators on certain H^p classes, Suppl. Rend. Circ. Mat. Palermo 1 (1981), 81–97.
21. E. M. Stein and S. Wainger, Problems in harmonic analysis related to curvature, Bull. Amer. Math. Soc. 84 (1978), 1239–1295.
22. E. M. Stein and G. Weiss, Introduction to Fourier Analysis on Euclidean Spaces, Princeton Univ. Press, Princeton, N.J., 1971.
23. J. -O. Strömberg, Non-equivalence between two kinds of conditions on weight functions, Proc. Symp. Pure Math. 35 part 1 (1979), 141–148.
24. J. -O. Strömberg and A. Torchinsky, Weighted Hardy spaces, Lecture Notes in Math. 1381, Springer-Verlag, Berlin Heidelberg New York London Paris Tokyo Hong Kong, 1989.
25. T. Tao, The weak-type endpoint Bochner-Riesz conjecture and related topics, Indiana Univ. Math. J. 47 (1998), 1097–1124.

Department of Mathematics, Faculty of Education, Kanazawa University, Kanazawa 920-1192, Japan

E-mail address: `shuichi@kenroku.kanazawa-u.ac.jp`